J. Fluid Mech. (1972), *vol.* 55, *part* 2, *pp.* 359–383
Printed in Great Britain

Two-dimensional boundary layers in a free stream which oscillates without reversing

By T. J. PEDLEY

Physiological Flow Studies Unit,† Imperial College, London, S.W. 7

(Received 13 December 1971)

The laminar boundary layer over the semi-infinite rigid plane $y = 0$, $x > 0$ is examined for the case when the free-stream velocity takes the form

$$U(x,t) = U_0(x)\,(1+\alpha_1 \sin \omega t),$$

where $0 \leqslant \alpha_1 < 1$ and $U_0(x) \propto x^n$, $0 \leqslant n \leqslant 1$. The corresponding steady solution is the Falkner–Skan boundary layer with zero ($n = 0$) or favourable pressure gradient ($n = 1$ corresponds to the stagnation-point boundary layer). The skin friction, and the heat transfer from the wall when that is maintained at a uniform temperature T_1 greater than the temperature T_0 of the oncoming fluid, are calculated by means of two asymptotic expansions: a regular one for small values of the frequency parameter $\epsilon_1(x) = \omega x/U_0(x)$ and a singular one (requiring the use of matched asymptotic expansions) for large values of ϵ_1. The principal difference between this work and that of earlier authors is that here α_1 is not required to be small. Numerical computations are made for three values of n $(0, \frac{1}{3}, 1)$, three values of α_1 (0·2, 0·5 and 0·8) and (in the case of the heat transfer) two values of the Prandtl number σ (0·72 and 7·1). It is demonstrated that for each n there is a value of ϵ_1 at which the small- and the large-ϵ_1 expansions for the skin friction overlap quite well, and that, near the overlap region, two terms of the small-ϵ_1 expansion provide a more accurate asymptotic representation of the solution than three. It is also shown that there is no region of overlap between the small- and the large-ϵ_1 heat-transfer expansions, except in one case ($n = \frac{1}{3}$, $\sigma = 0{\cdot}72$) where the overlap value of ϵ_1 ($= 2{\cdot}0$) is the same as for the skin-friction expansions. The question of the existence of eigensolutions in the large-ϵ_1 expansion, where on physical grounds they can be expected to appear, is discussed in the appendix.

1. Introduction

The problem to be considered in this paper is that of the laminar two-dimensional boundary layer which is generated on a semi-infinite plane boundary when a viscous incompressible fluid flows over it in such a way that the velocity on that boundary in the absence of viscosity has the form

$$U(x,t) = U_0(x)\,V(\omega t). \tag{1.1}$$

Here x is the distance measured along the boundary from its leading edge, t

† Also Department of Mathematics.

is the time and $1/\omega$ is a typical time scale of the motion. U_0 is restricted to the form

$$U_0(x) = Kx^n, \tag{1.2}$$

where K is a constant and $0 \leqslant n \leqslant 1$, and thus represents flow with favourable (or zero) pressure gradient over a wedge. The extremes $n = 0$ and $n = 1$ refer to flow over a flat plate and near a stagnation point respectively. The only restriction to be imposed at this stage on $V(\omega t)$ is that it should be always positive (no reversal of the free stream), although all numerical results will be calculated for the case in which

$$V(\omega t) = 1 + \alpha_1 \sin \omega t, \quad 0 \leqslant \alpha_1 < 1. \tag{1.3}$$

All solutions are then expected to be periodic in time; indefinitely growing (secular) solutions are rejected. In addition to investigating the viscous boundary layer, we examine the thermal boundary layers generated when the plane boundary is maintained at a uniform temperature T_1 different from the temperature T_0 of the oncoming fluid.

This problem has attracted the attention of a number of authors over the years. Lighthill (1954) treated the case of an arbitrary function $U_0(x)$, with time dependence (1.3), but with small values of the amplitude parameter α_1. He solved the first-order (in α_1) equations approximately in the two limiting cases where the frequency parameter

$$\epsilon_1(x) = \omega x / U_0(x) \tag{1.4}$$

tends to zero and infinity respectively, taking the first term in the asymptotic expansion at each limit. For small ϵ_1, Lighthill calculated the increase in amplitude and the phase lead of the skin friction over its value when $\epsilon_1 = 0$ (the quasi-steady case), and the corresponding decrease in amplitude and phase lag of the heat transfer when the wall is heated. For large ϵ_1, he showed that the asymptotic value of the phase lead of the skin friction is $\frac{1}{2}\pi$ and the phase lag of the heat transfer is $\frac{1}{4}\pi$. However, in the limit of large ϵ_1 the unsteady part of the boundary layer is confined to a thin Stokes layer, of thickness $(\nu/\omega)^{\frac{1}{2}}$, embedded within the steady boundary layer of the mean flow, of thickness $(\nu x/U_0)^{\frac{1}{2}}$, so that, as $\epsilon_1 \to \infty$ for fixed α_1, the velocity fluctuations are no longer small compared with the local mean flow in the Stokes layer. Thus the small-α_1 approximation is not uniformly valid as $\epsilon_1 \to \infty$.

Subsequent authors have extended Lighthill's solution, but in general have retained the small-α_1 approximation. Gibellato (1955) and Ghosh (1961) restricted themselves to the case of the semi-infinite flat plate ($n = 0$), and independently extended the small-ϵ_1 expansion (still to the first order in α_1) to several terms. Ghosh also extended the large-ϵ_1 expansion, but considered only the viscous, not the thermal, boundary layer. Rott & Rosenzweig (1960) did the same for the general Falkner–Skan boundary layer ($U_0(x)$ of the form (1.2) for any n), paying particular attention to the stagnation-point case ($n = 1$). Lam & Rott (1960) gave an exhaustive mathematical treatment of the linear first-order (in α_1) problem, including numerical calculations of the first 15 terms of the small-ϵ_1 expansion. Finally, Gersten (1965) repeated the small- and large-ϵ_1 expansions to the second order in α_1, in order to examine the interaction between the oscillatory boundary layer and the mean flow. The small amplitude approximation has been shown to be unnecessary in both the small-ϵ_1 case (see Moore (1951, 1957), who made most of the calculations contained in § 2 below) and the large-ϵ_1 case (Lin 1956; Gibson 1957) but this work has not been followed up except in a recent paper by Ishigaki (1970), who considered the case of stagnation-point flow ($n = 1$).

The main purpose of the present paper, then, is to solve the problem with no restriction on the value of α_1 except that it must be less than 1, for if $\alpha_1 \geqslant 1$ the boundary-layer approximation breaks down, at least near the leading edge of the semi-infinite boundary. In § 2 the small-ϵ_1 expansion is outlined, and the solutions for skin friction and heat transfer as functions of time are derived; this is a regular expansion. The large-ϵ_1 limit is singular, and the method of matched asymptotic expansions is used in § 3 to derive the first three terms for the skin friction and heat transfer. The problems which arise are similar, but rather more complicated, than those of the companion paper (Pedley 1972). In § 4 some numerical calculations of skin friction and heat transfer are made for three values of n, $n = 0$ (flat plate), $\frac{1}{3}$ (wedge of semi-vertex angle $\frac{1}{4}\pi$) and 1 (stagnation point), for three values of α_1 (0·2, 0·5 and 0·8) and for two values of the Prandtl number σ (0·72 ,the value for air, and 7·1, the value for water at 20 °C). The small- and large-ϵ_1 expansions for the skin friction are shown to overlap quite well for a certain range of values of ϵ_1 (different for each n), and between them should provide a good representation of the solution for all ϵ_1. Good overlap is not achieved for the heat-transfer expansions.

Let the origin of co-ordinates be at the leading edge of the semi-infinite plane boundary, with x measured along it and y at right-angles to it, and let the velocity components (u, v) be expressed in terms of a stream function ψ:

$$(u, v) = (\psi_y, -\psi_x). \tag{1.5}$$

Then the equation governing incompressible laminar viscous flow in the boundary layer is

$$\psi_{yt} + \psi_y \psi_{xy} - \psi_x \psi_{yy} = U_t + UU_x + \nu \psi_{yyy}, \tag{1.6}$$

where ν is the kinematic viscosity of the fluid and $U(x,t)$ is given by (1.1). The boundary conditions on ψ are

$$\psi = \psi_y = 0 \quad \text{on} \quad y = 0, \quad \psi_y \overset{e}{\to} U(x,t) \quad \text{as} \quad y \to \infty, \tag{1.7}$$

where $\overset{e}{\to}$ means 'tends exponentially to'. The thermal boundary-layer equation, expressed in terms of the dimensionless temperature $\theta = (T - T_0)/(T_1 - T_0)$, is

$$\theta_t + \psi_y \theta_x - \psi_x \theta_y = (\nu/\sigma)\, \theta_{yy}, \tag{1.8}$$

where σ is the Prandtl number. The boundary conditions on θ are

$$\theta = 1 \quad \text{on} \quad y = 0, \quad \theta \overset{e}{\to} 0 \quad \text{as} \quad y \to \infty. \tag{1.9}$$

These equations and boundary conditions govern the problem. The skin friction at the wall (divided by the viscosity) is

$$S(x,t) = \psi_{yy}|_{y=0} \tag{1.10}$$

and the heat transfer per unit area is

$$Q(x,t) = (-\mu/\sigma)\,(T_1 - T_0)\, \theta_y|_{y=0}. \tag{1.11}$$

2. Asymptotic expansion for small $\epsilon_1(x)$

The limit $\epsilon_1 \to 0$ is clearly the quasi-steady limit, and the steady non-dimensionalization is applicable. Introduce new variables τ, η' and ϕ defined by

$$t = \frac{\tau}{\omega}, \quad y = \left[\frac{2\nu x}{U(x,t)}\right]^{\frac{1}{2}} \eta', \quad \psi = [2\nu x U(x,t)]^{\frac{1}{2}} \phi(x, \eta', \tau), \tag{2.1}$$

where it is not necessary to non-dimensionalize x, since, as we shall see, it appears only in the dimensionless quantity $\epsilon_1(x)$ (definition (1.4)). When $U(x,t)$ is given by (1.1) and (1.2), but not yet (1.3), the dimensionless version of (1.6) is

$$\phi_{\eta'\eta'\eta'} + (n+1)\phi\phi_{\eta'\eta'} - 2n\phi_{\eta}^{2'} + 2x(\phi_x\phi_{\eta'\eta'} - \phi_{\eta'}\phi_{x\eta'}) - \epsilon_1(x)$$
$$\times\left[\frac{2}{V}\phi_{\eta'\tau} + \frac{2\dot{V}}{V^2}\phi_{\eta'} + \frac{\dot{V}}{V^2}\eta'\phi_{\eta'\eta'}\right] = -2n - 2\epsilon_1(x)\dot{V}/V^2, \tag{2.2}$$

where $\dot{V}$ is the first derivative of $V(\tau)$. The boundary conditions on ϕ are

$$\phi = \phi_{\eta'} = 0 \quad \text{on} \quad \eta' = 0, \quad \phi_{\eta'} \overset{e}{\to} 1 \quad \text{as} \quad \eta' \to \infty.$$

We seek a similarity solution in powers of $\epsilon_1(x)$ of the form

$$\phi = \phi_0(\eta') + \sum_{m=1}^{\infty} \epsilon_1^m(x)\,\phi_m(\eta', \tau).$$

If we substitute this into (2.2) (remembering that $\epsilon_1(x) \propto x^{1-n}$) and equate like powers of ϵ_1, we obtain a series of equations for the functions ϕ_m. The zero-order equation is

$$\phi_{0\eta'\eta'\eta'} + (n+1)\phi_0\phi_{0\eta'\eta'} + 2n(1 - \phi_{0\eta'}^2) = 0, \tag{2.3}$$

which, with the boundary conditions $\phi_0(0) = \phi_0'(0) = 0$, $\phi_0(\infty) = 1$, defines the steady Falkner–Skan boundary-layer problem (Rosenhead 1963, p. 235). Let us write the solution as $\phi_0 \equiv f_0(\eta')$; this is a function whose properties are well-known, and for the three values of n in which we are most interested ($0, \frac{1}{3}, 1$) it is tabulated in Rosenhead (1963). For a more detailed tabulation, the equation was re-integrated numerically, by means of the procedure outlined in Rosenhead, and the tables given there were used as a check. Those values of the function to be used later are given in table 1. The first- and second-order equations can be solved by functions ϕ_1 and ϕ_2 of the following form:

$$\phi_1 = \frac{\dot{V}}{V^2} f_{11}(\eta'), \quad \phi_2 = \frac{\dot{V}^2}{V^4} f_{21}(\eta') + \frac{\ddot{V}}{V^3} f_{22}(\eta'). \tag{2.4}$$

n	$\alpha_2 = f_0''(0)$	$\beta = \lim_{\eta\to\infty} (\eta - f_0)$
0	0·46960	1·21678
$\frac{1}{3}$	1·07119	0·69676
1	1·74314	0·45813

TABLE 1. The properties of $f_0(\eta)$

n	$f''_{11}(0)$	$f''_{21}(0)$	$f''_{22}(0)$	σ	$g'_{01}(0)$	$g'_{11}(0)$	$g'_{21}(0)$	$g'_{22}(0)$
0	1·19999	0·38308	−0·66424	0·72	−0·41809	0·02887	−0·09328	0·17704
				7·1	−0·91784	0·31927	0·14638	0·24267
$\frac{1}{3}$	0·62571	0·11756	−0·11827	0·72	−0·54914	0·17537	0·15886	−0·07640
				7·1	−1·26834	0·72892	1·16299	−0·60815
1	0·40373	0·04122	−0·03580	0·72	−0·70914	0·24844	0·35916	−0·16295
				7·1	−1·67500	0·96286	2·12980	−0·99153

TABLE 2

The functions $f_{mk}(\eta')$ satisfy inhomogeneous equations of the form

$$f'''_{mk}+(n+1)f_0 f''_{mk}-2[m(1-n)+2n]\,f'_0 f'_{mk} \\ +[2m(1-n)+1+n]f''_0 f_{mk}=\mathscr{F}_{mk}(\eta') \quad (2.5)$$

together with homogeneous boundary conditions; here $\mathscr{F}_{mk}$ depends only on f_0 and f_{lk}, where $l<m$. These linear equations have been solved numerically by the Runge–Kutta–Gill method, with values of $f_0(\eta')$ and its derivatives obtained by Lagrangian five-point interpolation from the tables of that function. The values of $f''_{mk}(0)$, required for the skin-friction calculation, are given for $n=0,\frac{1}{3},1$ in table 2. The procedure for calculating further terms in the small-ϵ_1 expansion is straightforward. Every term can be written as the sum of products of known functions of τ and functions of η' which can be computed from a knowledge of the previous functions, as in the expression for ϕ_2 in (2.4). It can be seen, incidentally, that this expansion must break down if $V(\tau)$ approaches zero, or if any of its derivatives are anywhere very large, because of the form of functions like ϕ_2. Hence the restriction that α_1 be less than 1.

The skin friction (1.10) can be written in dimensionless form as

$$S_1(x,\tau)=\left(\frac{2\nu x}{U_0}\right)^{\frac{1}{2}}\frac{S}{U_0}=V^{\frac{3}{2}}(\tau)\,\phi_{\eta'\eta'}|_{\eta'=0}$$

$$=V^{\frac{3}{2}}(\tau)\left\{f''_0(0)+\epsilon_1\frac{\dot V}{V^2}f''_{11}(0)+\epsilon_1^2\left[\frac{\dot V^2}{V^4}f''_{21}(0)+\frac{\ddot V}{V^3}f''_{22}(0)\right]+O(\epsilon_1^3)\right\}. \quad (2.6)$$

The heat equation (1.8) can be solved in a similar way. The same non-dimensionalization leads to the following equation for θ:

$$\frac{1}{\sigma}\theta_{\eta'\eta'}+(n+1)\,\phi\theta_{\eta'}+2x\,(\phi_x\theta_{\eta'}-\phi_{\eta'}\theta_x)=\frac{2\epsilon_1(x)}{V(\tau)}\left(\theta_\tau+\frac{\dot V}{2V}\eta'\theta_{\eta'}\right). \quad (2.7)$$

We try solutions of the form

$$\theta=g_{01}(\eta')+\epsilon_1\frac{\dot V}{V^2}g_{11}(\eta')+\epsilon_1^2\left[\frac{\dot V}{V^4}g_{21}(\eta')+\frac{\ddot V}{V^3}g_{22}(\eta')\right]+\dots$$

and obtain equations for the $g_{mk}(\eta')$ as follows:

$$(1/\sigma)\,g''_{mk}+(n+1)f_0 g'_{mk}-2m(1-n)f'_0 g_{mk}=\mathscr{G}_{mk}(\eta'), \quad (2.8)$$

where $\mathscr{G}_{mk}$ depends on $f_0, f_{lk}\ (l\leqslant m)$ and $g_{lk}\ (l<m)$; $\mathscr{G}_{01}(\eta')\equiv 0$. These are solved

numerically, by the same method as that for equations (2.5), subject to the boundary conditions

$$g_{01}(0) = 1, \quad g_{mk\ (m>0)}(0) = g_{mk\ (m\geqslant 0)}(\infty) = 0.$$

The values of $g'_{mk}(0)$, needed for the heat transfer, are given in table 2 for the two chosen values of σ.

The heat transfer per unit area (1.11) is given in dimensionless form as

$$\begin{aligned} Q_1(x,\tau) &= \left(\frac{2\nu x}{U_0}\right)^{\frac{1}{2}} \frac{Q}{\mu(T_1-T_0)} = -\frac{V^{\frac{1}{2}}(\tau)}{\sigma}\,\theta_{\eta'}\big|_{\eta'=0} \\ &= -\frac{V^{\frac{1}{2}}(\tau)}{\sigma}\left\{g'_{01}(0) + \epsilon_1 \frac{\dot{V}}{V^2} g'_{11}(0) + \epsilon_2\left[\frac{\dot{V}^2}{V^4} g'_{21}(0) + \frac{\ddot{V}}{V^3} g'_{22}(0)\right] + O(\epsilon_1^3)\right\}. \end{aligned} \tag{2.9}$$

3. Asymptotic expansion for large $\epsilon_1(x)$

Here $V(\tau)$ is restricted to the form (1.3), and the method of matched asymptotic expansions is used to solve the governing equations. Suitable outer and inner variables for the solution of (1.6) are, respectively (cf. § 4 of the companion paper),

Outer variables: $$\eta = \left[\frac{U_0(x)}{2\nu x}\right]^{\frac{1}{2}} y, \quad \tilde{\psi}(x,\eta,\tau) = [2\nu x U_0(x)]^{-\frac{1}{2}}\psi, \tag{3.1}$$

Inner variables:

$$\zeta = \left(\frac{\omega}{2\nu}\right)^{\frac{1}{2}} y = \frac{\eta}{\gamma_1(x)}, \quad \Psi(x,\zeta,\tau) = \left(\frac{\omega}{2\nu}\right)^{\frac{1}{2}} U_0^{-1}(x)\,\psi = \frac{\tilde{\psi}}{\gamma_1}, \tag{3.2}$$

where $$\gamma_1(x) = \epsilon_1^{-\frac{1}{2}}(x), \tag{3.3}$$

which is the small parameter in powers of which we shall expand the solutions for $\tilde{\psi}$ and Ψ. The lengths with which y is non-dimensionalized are the thicknesses of the steady (Falkner–Skan) and oscillatory (Stokes) boundary layers respectively.

3.1. *The outer expansion*

In terms of the outer variables, (1.6) becomes

$$\begin{aligned} 2\tilde{\psi}_{\eta\tau} - \gamma_1^2(x)\{\tilde{\psi}_{\eta\eta\eta} + (n+1)\,\tilde{\psi}\tilde{\psi}_{\eta\eta} - 2n\tilde{\psi}_\eta^2 + 2x(\tilde{\psi}_x\tilde{\psi}_{\eta\eta} - \tilde{\psi}_{x\eta}\tilde{\psi}_\eta)\} \\ = 2\alpha_1\cos\tau + 2\gamma_1^2(x)\,n(1+\alpha_1\sin\tau)^2. \end{aligned} \tag{3.4}$$

The outer boundary condition is that

$$\tilde{\psi}_\eta \overset{e}{\to} 1 + \alpha_1 \sin\tau \quad \text{as} \quad \eta\to\infty, \tag{3.5}$$

while the inner condition is that of matching to the inner solution (the inner condition on ψ is given by (1.7)). There is a further condition which states that, as $\alpha_1 \to 0$, ψ would tend to become the stream function for steady flow, that is

$$\tilde{\psi} \sim f_0(\eta) \quad \text{as} \quad \alpha_1 \to 0. \tag{3.6}$$

We seek an expansion for $\tilde{\psi}$ of the form

$$\tilde{\psi} = \sum_{m=0}^{\infty} \gamma_1^m(x)\,\tilde{\psi}_m(\eta,\tau), \tag{3.7}$$

which we substitute into (3.4), equating like powers of γ_1, to obtain a series of equations for the functions $\tilde{\psi}_m$. The zero-order equation is

$$\tilde{\psi}_{0\eta\tau} = \alpha_1 \cos\tau,$$

of which the general solution satisfying conditions (3.5) and (3.6) is

$$\tilde{\psi}_0 = \alpha_1 \eta \sin\tau + f_0(\eta) + F_0(\eta) + \dot{T}_0(\tau), \tag{3.8}$$

where the last two terms tend to zero as $\alpha_1 \to 0$ but are otherwise arbitrary functions. $\dot{T}_0(\tau)$ is expected to be oscillatory with zero mean; any constant part can be incorporated into F_0. The solution of the first-order equation is just

$$\tilde{\psi}_1 = F_1(\eta) + \dot{T}_1(\tau), \tag{3.9}$$

where F_1 and $\dot{T}_1$ are arbitrary functions. Here a dot and a prime denote differentiation with respect to τ and η respectively. Henceforth the τ dependence of functions will always be stated explicitly, but the η dependence will usually be understood.

The equation for $\tilde{\psi}_2$ is

$$\begin{aligned} 2\tilde{\psi}_{2\eta\tau} = \bar{f}_0''' + (n+1)\,[\eta\alpha \sin\tau + \bar{f}_0' + \dot{T}_0(\tau)]\bar{f}_0'' \\ - 2n(\bar{f}_0' + \alpha_1 \sin\tau)^2 + 2n(1+\alpha_1\sin\tau)^2, \end{aligned} \tag{3.10}$$

where we have used (3.8), putting $\bar{f}_0 \equiv f_0 + F_0$. In order that there should be no secular terms, $\bar{f}_0(\eta)$ must satisfy (2.3). Now this is the equation satisfied by $f_0(\eta)$, which also satisfies both the wall boundary conditions and the steady part of the boundary condition far from the wall, but we cannot yet rule out the existence of functions $F_0(\eta)$ such that $F_0(0)$ or $F_0'(0)$ is non-zero. After removing the time-independent part of (3.10), we can integrate that equation to give

$$\begin{aligned} \tilde{\psi}_2 = -\alpha_1 \cos\tau[\tfrac{1}{2}(n+1)\,\eta\bar{f}_0' - \tfrac{1}{2}(5n+1)\bar{f}_0 + 2n\eta] \\ + \tfrac{1}{2}(n+1)\bar{f}_0' T_0(\tau) + F_2(\eta) + \dot{T}_2(\tau), \end{aligned} \tag{3.11}$$

where again the last two terms are arbitrary functions. In all cases we expect the arbitrary functions occurring in $\tilde{\psi}_m$ to be determined by the equation obtained from prohibiting secular terms in $\tilde{\psi}_{m+2}$ and by boundary conditions derived from the inner expansion. Eigenfunctions may occur (solutions which satisfy both the outer and the wall boundary conditions identically), and there is nothing we can do to determine them.

The problems for $\tilde{\psi}_3$ and $\tilde{\psi}_4$ can be solved similarly. For secular terms to be absent, $F_1(\eta)$ and $F_2(\eta)$ must satisfy the equations

$$\begin{aligned} F_m''' + (n+1)\bar{f}_0 F_m'' + [m(1-n) - 4n]\bar{f}_0' F_m' - [m(1-n) - (1+n)]\bar{f}_0'' F_m \\ = -\delta_{m2}[2nF_1F_1'' + (1-3n)\,F_1'^2] \end{aligned} \tag{3.12}$$

$(m = 1, 2)$ and the boundary conditions

$$F_m' \overset{e}{\to} 0 \quad \text{as} \quad \eta \to \infty.$$

The solution for $\tilde{\psi}_3$ is

$$\begin{aligned} \tilde{\psi}_3 = -\alpha_1 \cos\tau[\tfrac{1}{2}(n+1)\,\eta F_1' - 3nF_1] + \tfrac{1}{2}(n+1)\,F_1' T_0(\tau) \\ + n\bar{f}_0' T_1(\tau) + \dot{T}_3(\tau) + F_3(\eta) \end{aligned} \tag{3.13}$$

and the solution for $\bar{\psi}_4$ is

$$\bar{\psi}_4 = -\tfrac{1}{2}\alpha_1 \sin\tau \Big\{\tfrac{1}{2}(1+n)\,\eta\bar{f}_0''' + \tfrac{1}{2}(1-3n)\bar{f}_0'' - 2n(1-3n)\,\eta\bar{f}_0' + 6n(1-3n)\bar{f}_0$$
$$+\tfrac{1}{2}(n+1)^2\,\eta\bar{f}_0\bar{f}_0'' - n(n+1)\,\eta\bar{f}_0'^2 + \tfrac{1}{2}(1-2n-19n^2)\bar{f}_0\bar{f}_0'$$
$$-\tfrac{1}{2}(1+9n)(1-5n)\int_0^\eta \bar{f}_0'^2(t)\,dt\Big\} + \tfrac{1}{8}\alpha_1^2\cos 2\tau$$
$$\times\{\tfrac{1}{2}(n+1)^2\eta^2\bar{f}_0'' + \tfrac{1}{2}(1+n)(1-11n)\eta\bar{f}_0' - \tfrac{1}{2}(1+5n)(1-7n)\bar{f}_0 + 4n(1-3n)\eta\}$$
$$-\tfrac{1}{2}\alpha_1\cos\tau\{(n+1)\,\eta F_2' + (1-7n)F_2\} - \tfrac{1}{2}(1-3n)\,T_2(\tau)\bar{f}_0'$$
$$+nT_1(\tau)\,F_1' + \dot{T}_4(\tau) + F_4(\eta), \tag{3.14}$$

where a number of terms containing $T_0(\tau)$ have been omitted because T_0 is subsequently shown to be zero. In each case, the last two terms are arbitrary functions. The outer boundary conditions on the terms in the *inner* expansion are obtained by rewriting the outer expansion (equations (3.8), (3.9), (3.11), (3.13) and (3.14)) in terms of the outer variable $\zeta = \eta/\gamma_1$ and expanding in powers of γ_1. Each term of that expansion can be used as the outer boundary condition on a corresponding term of the inner solution, as in the companion paper.

3.2. *The inner expansion*

In terms of inner variables, (1.6) becomes

$$2\Psi_{\zeta\tau} - \Psi_{\zeta\zeta\zeta} + 2\gamma_1^2(x)\{n\Psi_\zeta^2 - n\Psi\Psi_{\zeta\zeta} + x\Psi_{x\zeta}\Psi_\zeta - x\Psi_x\Psi_{\zeta\zeta}\}$$
$$= 2\alpha_1\cos\tau + 2n\gamma_1^2(x)\,(1+\alpha_1\sin\tau)^2, \tag{3.15}$$

with inner boundary condition

$$\Psi = \Psi_\zeta = 0 \quad \text{at} \quad \zeta = 0. \tag{3.16}$$

We seek a solution of the form

$$\Psi = \sum_{m=-1}^{\infty} \gamma_1^m(x)\,\Psi_m(\zeta,\tau),$$

recalling that $\bar{\psi} = \gamma_1\Psi$, and substitute this series into (3.15), equating like powers of γ_1.

The problem for Ψ_{-1} is then

$$2\Psi_{-1\zeta\tau} - \Psi_{-1\zeta\zeta\zeta} = 0; \quad \Psi_{-1}(0,\tau) = \Psi_{-1\zeta}(0,\tau) = 0, \quad \Psi_{-1}(\infty,\tau) = F_0(0) + \dot{T}_0(\tau),$$

which has no steady or periodic (non-diffusing) solution unless $\Psi_{-1}(\infty,\tau) = 0$, in which case $\Psi_{-1} \equiv 0$. Thus, without loss of generality, we may set

$$T_0(\tau) \equiv 0, \quad F_0(0) = 0. \tag{3.17}$$

The problem for Ψ_0 is

$$2\Psi_{0\zeta\tau} - \Psi_{0\zeta\zeta\zeta} = 2\alpha_1\cos\tau,$$
$$\Psi_0(0,\tau) = \Psi_{0\zeta}(0,\tau) = 0, \quad \Psi_0(\infty,\tau) \sim F_1(0) + \dot{T}_1(\tau) + \alpha_1\zeta\sin\tau + \zeta F_0'(0),$$

where the last condition comes from (3.8) and (3.9). This has no periodic solution unless

$$F_0'(0) = F_1(0) = 0, \quad T_1(\tau) \equiv \tfrac{1}{2}\alpha_1(\sin\tau + \cos\tau), \tag{3.18}$$

and then

$$\Psi_0 = \alpha_1 \zeta \sin\tau - \mathrm{Im}\left\{\frac{\alpha_1 e^{i\tau}}{1+i}\left[1 - e^{-(1+i)\zeta}\right]\right\}. \tag{3.19}$$

This represents the Stokes layer, whose existence in this problem was first demonstrated by Lighthill (1954) and Lin (1956). Note that we now have three boundary conditions on the function $F_0(\eta)$, where $\bar{f}_0 = f_0 + F_0$ satisfies (2.3); these are

$$F_0(0) = F_0'(0) = F_0'(\infty) = 0.$$

Thus $F_0(\eta)$ will be identically zero unless $f_0(\eta)$ is not the unique solution of the Falkner–Skan problem. The large amount of analytical and numerical work which has been done on this problem indicates that $f_0(\eta)$ is the unique solution, so that no eigenfunction exists. The analysis in the appendix proves that there can be no eigensolution for small α_1, i.e. $F_0 \ll f_0$, for in that case (2.3) could be linearized to give

$$F_0''' + (n+1) f_0 F_0'' - 4n f_0' F_0' + (n+1) f_0'' F_0 = 0, \tag{3.20}$$

which is of the form (A 1), with $\lambda = -4n$. We may therefore set $\bar{f}_0(\eta) \equiv f_0(\eta)$.

Subsequent terms in the inner expansion, and boundary conditions on the unknown terms of the outer expansion, are obtained similarly, but with increasingly lengthy algebra. We require the expansion

$$f_0(\eta) = \tfrac{1}{2}\alpha_2 \eta^2 - \tfrac{1}{3} n \eta^3 - \frac{(1-3n)}{5\,!}\alpha_2^2 \eta^5 + \ldots,$$

where $\alpha_2 = f_0''(0)$ is given for $n = 0$, $\frac{1}{3}$ and 1 in table 1. The results for the next three terms of the inner expansion are given below.

$$\Psi_1 = \tfrac{1}{2}\alpha_2 \zeta^2, \tag{3.21}$$

requiring

$$F_1'(0) = F_2(0) = T_2(\tau) = 0. \tag{3.22}$$

$$\begin{aligned}\Psi_2 = {} & \tfrac{13}{8} n\alpha_1^2 - \tfrac{3}{4} n\alpha_1^2 \zeta + \tfrac{1}{2} F_1''(0)\,\zeta^2 - \tfrac{1}{3} n \zeta^3 - \tfrac{1}{8} n\alpha_1^2 e^{-2\zeta} - \tfrac{1}{2} n\alpha_1^2 e^{-\zeta} \\ & \times \{3\cos\zeta + (\zeta+2)\sin\zeta\} + \mathrm{Im}\{e^{i\tau}[-2n\alpha_1 i\zeta + n\alpha_1 k(1-e^{-k\zeta})] \\ & + e^{2i\tau}[\tfrac{1}{2} n\alpha_1^2 \zeta e^{-k\zeta} - (n\alpha_1^2 \bar{k}/4\sqrt{2})(1 - e^{-\sqrt{2}k\zeta})]\},\end{aligned} \tag{3.23}$$

where $k = 1+i$, $\bar{k} = 1-i$, and we require

$$\left.\begin{aligned} F_3(0) = \tfrac{13}{8} n\alpha_1^2, \quad F_2'(0) = -\tfrac{3}{4} n\alpha_1^2, \\ T_3(\tau) = \mathrm{Im}\{n\alpha_1 \bar{k}\, e^{i\tau} + (n\alpha_1^2/8\sqrt{2})\, k\, e^{2i\tau}\}. \end{aligned}\right\} \tag{3.24}$$

Finally,

$$\begin{aligned}\Psi_3 = {} & \tfrac{1}{2} F_2''(0)\,\zeta^2 + \mathrm{Im}\{\alpha_1\alpha_2 e^{i\tau}[\tfrac{1}{32}(13-75n) + \tfrac{1}{2} nk\zeta - \tfrac{1}{4}(1-3n)\, i\zeta^2 \\ & - e^{-k\zeta}(\tfrac{1}{32}(13-75n) + \tfrac{1}{32}(13-59n)\, k\zeta + \tfrac{1}{16}(5-19n)\, i\zeta^2 - \tfrac{1}{24}(1-3n)\,\bar{k}\zeta^3)]\},\end{aligned} \tag{3.25}$$

where we require

$$\left.\begin{aligned} F_3'(0) = F_4(0) = 0, \\ T_4(\tau) = -\tfrac{3}{32}\alpha_1\alpha_2(7-33n)\cos\tau. \end{aligned}\right\} \tag{3.26}$$

Note that all the boundary conditions for the functions F_1, F_2 and F_3 have now been obtained. Unless there is an eigensolution, F_1 will be identically zero, but this is true for F_2 and F_3 only in the case $n = 0$. In the appendix, the eigenvalue problems for these functions are examined in detail. It is proved there that no eigensolutions to the problems determining the functions $F_m(\eta)$, for any $m \geq 0$, exist in the case $n = 1$ and that no eigensolutions exist if $m \leq 4$ in the case $n = \frac{1}{3}$. In the case $n = 0$, however, the first eigenvalue is $m = 2$, and in that case the function $F_2(\eta)$ is an arbitrary multiple of $f_0 - \eta f_0'$.

On physical grounds, the appearance of eigenvalues for $n < 1$ is to be expected in order to take account of different upstream conditions in the boundary layer. If there were no eigenvalues, the large-ϵ_1 expansion, in inverse powers of $x^{\frac{1}{2}(1-n)}$, would be completely determined, and the flow at large x would be independent of that at small x (except for $n = 1$, where the expansion is independent of x), although the boundary-layer equations are parabolic in x. In general, all eigensolutions, corresponding to both integer and non-integer eigenvalues m, will be required to take account of upstream conditions, although only the former will affect the above expansion. In this paper, the expansion is terminated at the $\gamma_1^3(x^{-\frac{3}{2}(1-n)})$ term, so the only eigensolution to influence the numerical results will be the function $F_2(\eta)$, corresponding to $m = 2$ in the case $n = 0$. The next eigensolution to appear would also be in the case $n = 0$, corresponding to $m = 3{\cdot}77$ (Libby & Fox 1963).

The eigensolutions considered in the above discussion are steady and influence the unsteady expansion only through the eigenfunctions corresponding to integer eigenvalues (e.g. $m = 2$ for $n = 0$). Unsteady eigensolutions have been considered by Lam & Rott (1960), and more recently by Ackerberg & Phillips (1972), in the small amplitude (first order in α_1), flat-plate ($n = 0$) case. They have demonstrated the existence of unsteady eigensolutions which decay exponentially with x and would not appear in an algebraic expansion in inverse powers of x. The effect of these eigensolutions on the skin friction is small for undisturbed upstream conditions, but can become large if a perturbation is introduced into the boundary layer (in Ackerberg & Phillips's numerical experiment, the perturbation was introduced at $\epsilon_1 = 1{\cdot}0$ and its effect was still discernible at $\epsilon_1 = 6{\cdot}0$). Similar unsteady eigensolutions are to be expected in the present problem, but their form will not be the same as those of Ackerberg & Phillips, because we here take the limit $\epsilon_1 \to \infty$ without first having taken the limit $\alpha_1 \to 0$. This problem is examined briefly in the appendix, and the probable form of the eigenfunctions, if they exist, is given; existence has not yet been proved. We may note the contrast between this problem and that of the companion paper, where an infinite set of eigenfunctions does appear in the algebraic expansion, and there are no unsteady eigensolutions, exponentially decaying in x.

If we ignore the exponentially decaying eigensolutions, we may now write down the large-ϵ_1 expansion for the skin friction S (from equation 1.10), which is given in dimensionless form by

$$\begin{aligned} S_2(x,\tau) &= \left(\frac{2\nu x}{U_0}\right)^{\frac{1}{2}} \frac{S}{U_0} = \gamma_1^{-1}(x)\Psi_{\zeta\zeta}|_{\zeta=0} \\ &= \gamma_1^{-1}\{\alpha_1(\cos\tau + \sin\tau) + \gamma_1\alpha_2 \\ &\quad + \gamma_1^2[\tfrac{1}{2}n\alpha_1^2 + 2n\alpha_1(\sin\tau - \cos\tau) - n\alpha_1^2(1 - 1/\surd 2)(\sin 2\tau + \cos 2\tau)] \\ &\quad + \gamma_1^3[F_2''(0) - \tfrac{1}{16}\alpha_1\alpha_2(5 - 19n)\cos\tau] + O(\gamma_1^4)\}. \end{aligned} \tag{3.27}$$

n	$C_2 = \alpha_1^{-2} F_2''(0)$	σ	$G_2'(0)$	$G_3'(0)$
0	?	0·72	0	0
		7·1	0	0
$\frac{1}{3}$	0·15536	0·72	−0·0001086	0·49721
		7·1	−0·0006378	11·4715
1	0·86051†	0·72	0·21636	−1·24370
		7·1	1·20715	−10·4457

† This agrees with the result of Ishigaki (1970) to 3 significant figures.

TABLE 3

The values of $F_2''(0)$ for $n = \frac{1}{3}$ and 1 were obtained by numerical integration of (3.12) with boundary conditions given by (3.22), (3.24) and $F_2 \to 0$ as $\eta \to \infty$, and are given by $F_2''(0) = \alpha_1^2 C_2$, where C_2 is tabulated in table 3. In all future computations, the indeterminate $F_2''(0)$ for $n = 0$ will be taken to be zero.

3.3. *Solution of the heat equation*

The large-ϵ_1 expansion for the temperature θ can now be developed in a similar way. Let the outer and inner representations of θ be $\tilde{\theta}$ and Θ respectively. In terms of outer variables the heat equation (1.8) is

$$2\tilde{\theta}_\tau = \gamma_1^2(x)\{(1/\sigma)\tilde{\theta}_{\eta\eta} + (1+n)\tilde{\psi}\tilde{\theta}_\eta + 2x(\tilde{\psi}_x\tilde{\theta}_\eta - \tilde{\psi}_\eta\tilde{\theta}_x)\}. \tag{3.28}$$

We seek a solution for $\tilde{\theta}$ in the form of an expansion in powers of γ_1 like (3.7) for $\tilde{\psi}$, which must reduce to the steady solution $g_{01}(\eta)$ (see equation (2.8)) in the limit $\alpha_1 \to 0$ for all γ_1, and all of whose terms must tend to zero exponentially at infinity. Solutions for the first few terms of the outer expansion are

$$\left.\begin{aligned}
\tilde{\theta}_0 &= g_0(\eta) + G_0(\eta) \equiv \bar{g}_0(\eta),\\
\tilde{\theta}_1 &= G_1(\eta),\\
\tilde{\theta}_2 &= -\tfrac{1}{2}(1+n)\,\alpha_1\eta\bar{g}_0'\cos\tau + G_2(\eta),\\
\tilde{\theta}_3 &= -\tfrac{1}{2}(1+n)\,\alpha_1\eta G_1'\cos\tau - \tfrac{1}{2}(1-n)\,\alpha_1 G_1\cos\tau\\
&\quad + \tfrac{1}{2}n\alpha_1\bar{g}_0'(\sin\tau+\cos\tau) + G_3(\eta),\\
\tilde{\theta}_4 &= \tfrac{1}{2}\alpha_1\sin\tau\{-[(1+n)/2\sigma](\eta\bar{g}_0''' + 2\bar{g}_0'') + 2n(1-3n)\eta\bar{g}_0' - \tfrac{1}{2}(1+n)^2\eta\\
&\quad\times(f_0\bar{g}_0'' + f_0'\bar{g}_0') - (1+2n-7n^2)f_0\bar{g}_0' + G_1'\} - \tfrac{1}{2}\alpha_1\cos\tau\\
&\quad\times\{(1+n)\eta G_2' + 2(1-n)\,G_2 - nG_1'\} + \tfrac{1}{4}\alpha_1^2\cos 2\tau\\
&\quad\times\{\tfrac{1}{4}(1+n)^2\eta^2\bar{g}_0'' + \tfrac{1}{4}(3-n)\,(1+n)\,\eta\bar{g}_0'\} + G_4(\eta),
\end{aligned}\right\} \tag{3.29}$$

where the functions $G_m(\eta)$ are functions of integration, which must satisfy certain second-order ordinary differential equations in order that no secular terms should appear. G_0 satisfies (2.8) with $m = 0$ and G_m ($m = 1, 2, 3$) satisfies the equation

$$(1/\sigma)\,G_m'' + (1+n)f_0'G_m' + m(1-n)f_0'G_m = \mathscr{H}_m(\eta), \tag{3.30}$$

where $\mathscr{H}_1 \equiv 0$ and $\mathscr{H}_{m>1}$ is known. It is shown in the appendix that the homogeneous parts of these equations do not have eigensolutions satisfying both the condition at infinity and the condition $G_m(0) = 0$, for the values of n, m and σ

in which we are interested. If unsteady eigensolutions, exponentially decaying in x, exist for the stream function, then they presumably also exist for the temperature.

In terms of inner variables, the heat equation is

$$2\Theta_\tau - (1/\sigma)\,\Theta_{\zeta\zeta} = 2\gamma_1^2\{n\Psi\Theta_\zeta - x(\Theta_x\Psi_\zeta - \Theta_\zeta\Psi_x)\}. \tag{3.31}$$

Again, we seek a solution in powers of γ_1; the boundary condition at the wall requires that $\Theta_0(0,\tau) = 1$, $\Theta_{m>0}(0,\tau) = 0$. The boundary condition as ζ tends to infinity is obtained, as usual, by replacing η by $\gamma_1\zeta$ in the outer solution (3.30), and once more expanding in powers of γ_1. The condition that the equations for the Θ_m should have non-diffusing solutions leads to definite values for the constants $G_m(0)$, so that the functions $G_m(\eta)$ are fully determined in the absence of eigensolutions. The results of this procedure are as follows, where $\beta_1 = g'_{01}(0)$ (given in table 2): $\Theta_0 \equiv 1$, $G_0(0) = 0$, so that $G_0(\eta) \equiv 0$ and $\bar{g}_0(\eta) \equiv g_{01}(\eta)$; $\Theta_1 \equiv \beta_1\zeta$, $G_1(0) = 0$, so that $G_1(\eta) \equiv 0$; $\Theta_2 \equiv 0$, $G_2(0) = 0$, but $G_2(\eta) \not\equiv 0$ because equation (3.30) with $m = 2$ is inhomogeneous ($\mathscr{H}_2(\eta)$ is in fact zero for $n = 0$ and $n = \frac{1}{3}$).

$$\begin{aligned}\Theta_3 &\equiv G'_2(0)\,\zeta + \alpha_1\beta_1 \operatorname{Im}\left\{e^{i\tau}\left[\frac{kn}{2} - \frac{i(1+n)}{2}\zeta + \frac{k(\sigma-n)}{2(1-\sigma)^2}e^{-k\zeta/\sqrt{\sigma}}\right.\right.\\ &\qquad \left.\left. - \frac{\sigma\, e^{-k\zeta}}{2(1-\sigma)}\left(\frac{k(1+n\sigma-2n)}{1-\sigma} + i(1-n)\,\zeta\right)\right]\right\} \quad (\sigma \neq 1),\\ &\equiv G'_2(0)\,\zeta + \alpha_1\beta_1 \operatorname{Im}\{e^{i\tau}[\tfrac{1}{2}kn - \tfrac{1}{2}i(1+n)\,\zeta\\ &\qquad - e^{-k\zeta}(\tfrac{1}{2}kn + \tfrac{1}{8}i(1+3n)\,\zeta + \tfrac{1}{8}(1-n)\,\bar{k}\zeta^2)]\} \quad (\sigma = 1),\\ &\qquad G_3(0) = 0, \quad \text{but} \quad G_3 \not\equiv 0.\\ \Theta_4 &\equiv G'_3(0)\,\zeta - (n+1)\,\alpha_2\beta_1\frac{\sigma\zeta^4}{4!}, \quad G_4(0) = 0.\end{aligned}$$

The values of $G'_2(0)$ and $G'_3(0)$, required in the above equations, have been computed where necessary from the numerical integration of equations (3.30), and are given in table 3 for $n = 0$, $\frac{1}{3}$ and 1. For the calculation of $G_2(\eta)$ in the case $n = 0$, the function $F_2(\eta)$ was taken to be identically zero.

The heat transfer per unit area, see (1.11), is given in dimensionless form as

$$\begin{aligned}Q_2(x,\tau) &= \left(\frac{2\nu x}{U_0}\right)^{\frac{1}{2}} \frac{Q}{\mu(T_1-T_0)} = -\frac{\epsilon_1^{\frac{1}{2}}(x)}{\sigma}\,\Theta_\zeta|_{\zeta=0}\\ &= -\frac{1}{\sigma}\left\{\beta_1 + \gamma_1^2\left(G'_2(0) - \alpha_1\beta_1\cos\tau\left\{\frac{(1+n)}{2} + \frac{(\sigma-n)}{(1-\sigma)^2\sqrt{\sigma}}\right.\right.\right.\\ &\qquad \left.\left.\left. + \frac{\sigma}{2(1-\sigma)}\left[1 - n - \frac{2(1+n\sigma-2n)}{1-\sigma}\right]\right\}\right) + \gamma_1^3 G'_3(0) + O(\gamma_1^4)\right\} \quad (\sigma \neq 1) \end{aligned} \tag{3.32}$$

from the above equations for the Θ_m, with a corresponding expression for $\sigma = 1$.

4. Numerical results and discussion

In this section we present the results of calculations of the skin friction and heat transfer, obtained from both the small-ϵ_1 and the large-ϵ_1 expansions. The primary purpose is to determine over what range of values of ϵ_1 the two expansions overlap, if any, especially for values of the amplitude parameter α_1 larger than those considered by other authors. Lighthill (1954), for instance, shows that the phase of the skin friction (calculated from two terms of the small- and large-ϵ_1 expansions, for small α_1, using the Karman–Pohlhausen method) coincides in the two expansions for values of ϵ_1 of 0·6 in the case of the flat-plate boundary layer ($n = 0$) and 5·6 in the case of the stagnation-point boundary layer ($n = 1$). The amplitudes of the skin-friction variations also approximately coincide at these values of ϵ_1. The corresponding amplitudes and phases of the heat transfer (for a Prandtl number $\sigma = 0{\cdot}7$) do not agree so well, but smooth curves can be drawn, linking the two expansions, in the case of the stagnation-point boundary layer. Other authors appear to find somewhat different overlap values of ϵ_1. For example, Ghosh (1961) postulates a value of 1·0 in the case $n = 0$, and in the same case, the results of Lam & Rott (1960) and Ackerberg & Phillips (1972), using 15 terms of the small-ϵ_1 expansion, show a value of about 1·6. In the case $n = 1$, Ishigaki (1970) agrees with Lighthill's value of about 5·6, but Gersten (1965) shows a value of about 3·0. Gersten's results show that the expansions for heat transfer ($\sigma = 0{\cdot}7$) do not even approximately overlap for either $n = 0$ or $n = 1$. We, too, consider the two extreme cases $n = 0$ and $n = 1$, together with the intermediate case $n = \frac{1}{3}$.

In the small amplitude, two-term expansions of Lighthill, consisting of the mean and one oscillatory term, agreement of the amplitude and phase of the skin friction (or heat transfer) variations is the obvious criterion for deciding the point of overlap of the two expansions. Here, with more terms, involving second harmonics, the criterion is less clearcut. We consider four quantities as possible candidates for the overlap criterion: the magnitude of the maximum skin friction, the phase of this maximum, the overall amplitude of the skin-friction variation, and the mean skin friction. These four quantities are plotted against ϵ_1 in figure 1 for the case $n = 0$ and $\alpha_1 = 0{\cdot}5$ (not small); the two solid curves in each case represent the small- and large-ϵ_1 expansions, as given by the quantities $S_1(x,\tau)$ (equation (2.6)) and $S_2(x,\tau)$ (equation (3.27)) respectively. We see that the values of the amplitude and the maximum from the two expansions never coincide; they approach most closely at values of ϵ_1 in the ranges 0·5–0·6 and 0·6–0·7, respectively. The means agree at $\epsilon_1 \approx 0{\cdot}47$, and the phase of the maximum at $\epsilon_1 = 0{\cdot}6$. These values are close together, suggesting that there is a region where the two expansions approximately overlap even at this value of α_1. We choose to define the overlap value of ϵ_1 as that value at which there is closest agreement for the maximum skin friction. The corresponding values of ϵ_1 in the cases $\alpha_1 = 0{\cdot}2$ and $\alpha_1 = 0{\cdot}8$ are given in table 4. To see how good the overlap is, we examine in detail the complete cycle, rather than single representative quantities. In figure 2 the quantities $S_1(x,\tau)$ (calculated from the three terms of (2.6)) and $S_2(x,\tau)$ (from equation (3.27)) are plotted against τ in the case $n = 0$, $\epsilon_1 = 0{\cdot}6$, $\alpha_1 = 0{\cdot}5$. Agreement is fairly good everywhere except near the minimum skin friction, where the small-ϵ_1 expansion gives values considerably lower than does the large-ϵ_1 expansion. This is typical of all our calculations, and reflects the fact that, near $\tau = \frac{3}{2}\pi$, the free-stream velocity is smallest, so that the instantaneous value of $\omega x/U(\tau)$ is greatest and the small-ϵ_1 expansion is most likely to break down there. The discrepancy becomes even more marked

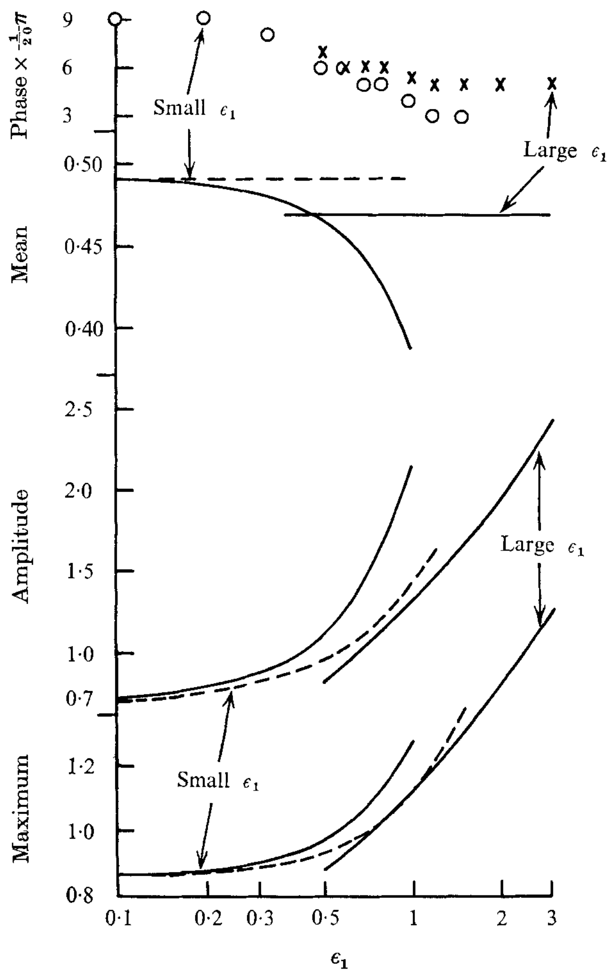

FIGURE 1. Comparison of the values of the maximum, the amplitude, the mean and the phase of the maximum of the skin-friction variation, calculated as functions of ϵ_1 from the small- and the large-ϵ_1 expansions (2.6) and (3.28) respectively. Broken line represents only two terms of (2.6). The case shown has $n = 0$, $\alpha_1 = 0{\cdot}5$.

n	0	0	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	1	1	1
α_1	0·2	0·5	0·8	0·2	0·5	0·8	0·2	0·5	0·8
ϵ_1	0·65†	0·65†	0·70†	2·4	2·1	2·2	6·2	6·6	6·0

† Coincidence does not occur in these cases and the value given is that of closest approach.

TABLE 4. Values of ϵ_1 at which the values of the maximum of the skin-friction variations, as calculated from the two expansions (2.6) and (3.27), coincide

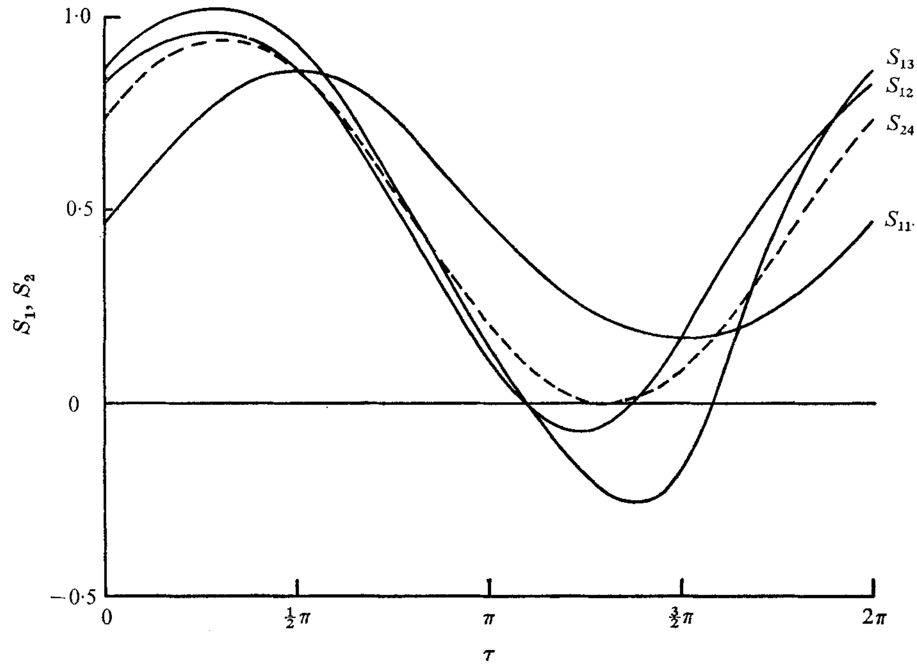

FIGURE 2. Graphs of $S_1(x, \tau)$ and $S_2(x, \tau)$ against τ as calculated from (2.6) and (3.27). S_{pq} represents the q-term expansion of S_p. ——, S_1; - - -, S_2. $n = 0$, $\alpha_1 = 0{\cdot}5$, $\epsilon_1 = 0{\cdot}6$.

for larger values of α_1, and it is for this reason that we choose the point of maximum skin friction as the point at which to determine overlap. Also shown in figure 2 are the curves of S_1 against τ as calculated from one and two terms of (2.6). These confirm that, except near the minimum, the difference between the three- and the two-term expansions is considerably smaller than that between the two- and the one-term expansions, so that the small-ϵ_1 expansion is a useful asymptotic expansion of the real solution for $\epsilon_1 \leqslant 0{\cdot}6$. Because of the error near the minimum, however, overall agreement between the small- and large-ϵ_1 expansions is better when only two terms, not three, of the former are used. This is again more marked for larger values of α_1 and indicates that the asymptotic expansion for small-ϵ_1 should be curtailed after two terms for greatest accuracy at values of ϵ_1 near overlap. This assumes that the four-term large-ϵ_1 expansion is itself close to the true solution; since the third term of (3.27) is zero for $n = 0$, a comparison of the two-, three- and four-term large-ϵ_1 expansions is not very informative, but a similar comparison for $n = \frac{1}{3}$ (figure 3) does indicate that the large-ϵ_1 expansion is a useful approximation near overlap. The broken lines in figure 1 show the maximum, the amplitude and the mean of the skin friction as calculated from only two terms of (2.6). Although the mean never coincides with the large-ϵ_1 mean, the curves of the maximum and the amplitude are brought much closer to their large-ϵ_1 counterparts, with closest approach at a somewhat larger value of ϵ_1 than before.

The other values of n $(=\frac{1}{3}, 1)$ have been considered in a similar way. In these cases, overlap does occur for both the maximum and the amplitude of the skin

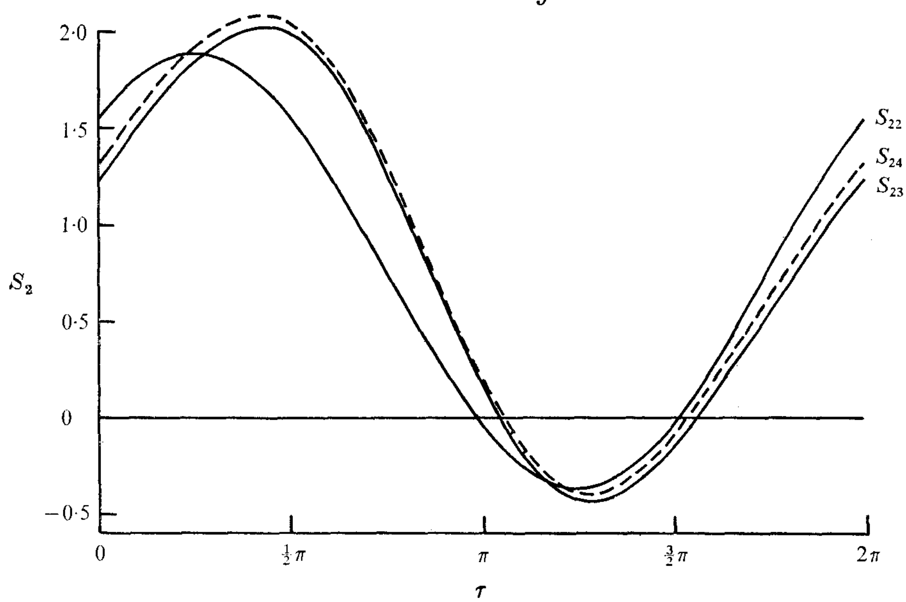

FIGURE 3. Graphs of $S_2(x, \tau)$ against τ as calculated from 2, 3 and 4 terms of (3.27) respectively. $n = \frac{1}{3}$, $\alpha_1 = 0{\cdot}8$, $\epsilon_1 = 2{\cdot}0$.

friction, but at different value of ϵ_1 (2·2 and 0·58 respectively, for example, in the case $n = \frac{1}{3}$, $\alpha_1 = 0{\cdot}8$). In graphs corresponding to those of figure 1, the curves of mean skin friction never cross, and those of the phase of the maximum are very close together for a range of values of ϵ_1. The overlap values for the maximum in these cases are all given in table 4; because this is likely to be the point of greatest accuracy of the small-ϵ_1 expansion, we choose $\epsilon_1 = 2{\cdot}0$ as the overlap value for the case $n = \frac{1}{3}$, and $\epsilon_1 = 6{\cdot}0$ in the case $n = 1$. Figure 3 shows that the difference between the four- and the three-term large-ϵ_1 expansion (for $n = \frac{1}{3}$, $\alpha_1 = 0{\cdot}8$) is both very small and small compared with that between the three- and the two-term large-ϵ_1 expansions; that is, this expansion is likely to be an accurate approximation to the true solution at this value of ϵ_1. The corresponding small-ϵ_1 expansions are given in figure 4, and again the three-term expansion breaks down near the point of minimum velocity. Also plotted is the curve of the four-term large-ϵ_1 expansion, and it can be seen to agree more closely with the two-term than with the three-term small-ϵ_1 expansion, at least over half the cycle. Agreement between the amplitude and the mean of the two expansions is also better with only two terms of the small-ϵ_1 expansion. At some lower value of ϵ_1, of course, the three-term small-ϵ_1 expansion must become more accurate than the two-term expansion, but then they are so close to each other that the latter can be used with confidence.

The $n = 1$ case yields similar results. At $\epsilon_1 = 6{\cdot}0$ the high-ϵ_1 expansion is seen to be useful for values of α_1 up to 0·8 (from graphs similar to those of figure 3), while the three-term low-ϵ_1 expansion breaks down near the point of minimum velocity, although in this case it agrees quite well with the large-ϵ_1 expansion for more than half the cycle. Over the rest of the cycle the two-term expansion agrees more closely, and is again to be preferred overall.

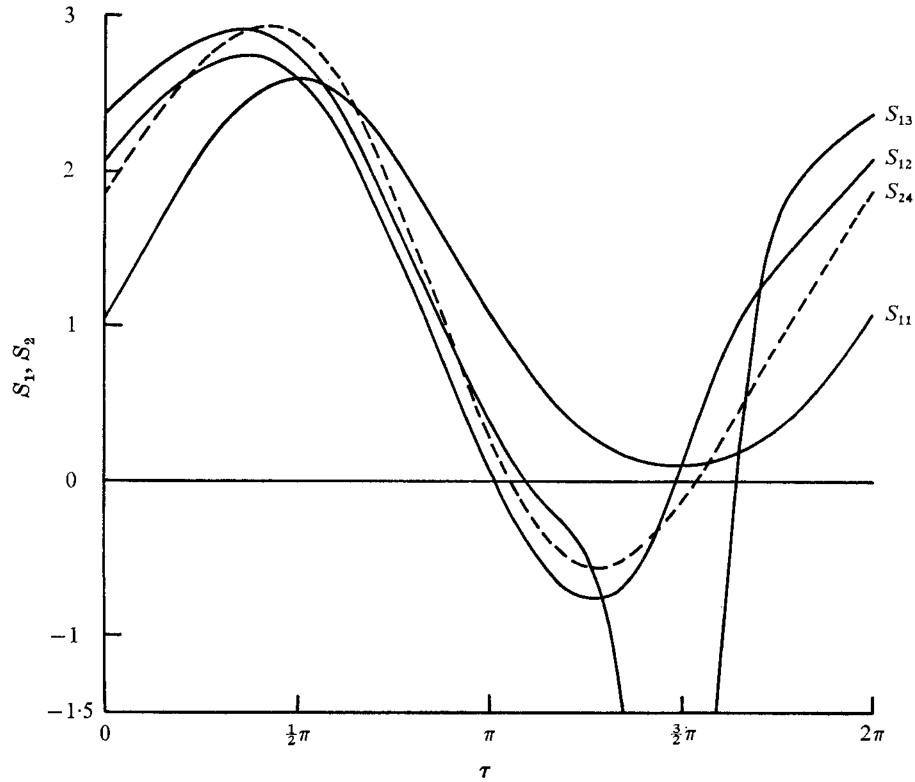

FIGURE 4. Graphs of $S_1(x, \tau)$ and $S_2(x, \tau)$ against τ as calculated from (2.6) and (3.27). $n = \frac{1}{3}$, $\alpha_1 = 0{\cdot}8$, $\epsilon_1 = 2{\cdot}0$. Other notation as in figure 2.

When we turn to the heat transfer, we find that overlap between the two expansions is far less satisfactory. We discuss the results for only one value of α_1 ($= 0{\cdot}5$), since the above results for skin friction (table 4) show that the value of α_1 has little effect on the overlap value of ϵ_1. The maximum, amplitude, mean and phase of the maximum of heat transfer have been calculated from the small- and large-ϵ_1 expansions from (2.9) and (3.32) respectively, for the same three values of n and for two values of the Prandtl number σ (0·72 and 7·1). Once more, the two-term small-ϵ_1 expansion is most appropriate for comparison with the large-ϵ_1 expansion, because of the large magnitude of the third term over part of the cycle. In this case, the two-term large-ϵ_1 expansion is also the most appropriate because, for $n = 0$, the third term is zero, and, for the other values of n, the quantity $G_3'(0)$ has a relatively large absolute value (table 3). The values of ϵ_1 at which overlap is closest, by the various criteria, are given in table 5; we see that they depend strongly on σ. At these 'overlap' values of ϵ_1, agreement between the two expansions is not as close as for skin friction. Figure 5 (a) shows the comparison over the whole cycle in the case $n = 0$, $\alpha_1 = 0{\cdot}5$, $\epsilon_1 = 0{\cdot}4$, $\sigma = 7{\cdot}1$ (the value of ϵ_1 was chosen as being close to the value given in table 5 for the maximum). The most marked difference between the two curves is in the phase. In only one case, $n = \frac{1}{3}$, $\sigma = 0{\cdot}72$, do the amplitude, the maximum *and the phase* all approximately coincide at $\epsilon_1 = 2{\cdot}0$, and figure 5(b) confirms that overlap is

n	σ	Maximum	Amplitude	Mean	Phase
0	0·72	6·0	5·0	None	7·0–8·0
	7·1	0·44	0·36	None	> 3·0
$\frac{1}{3}$	0·72	2·5	2·1	None	2·1
	7·1	0·36	0·35	0·54	None
1	0·72	< 0·5	0·92	None	None
	7·1	3·2	0·5	None	None

TABLE 5. Values of ϵ_1 at which apparent overlap occurs between the two heat-transfer expansions, as calculated from two terms of the expansions (2.9) and (3.32). 'None' means that the two expansions give exactly parallel results. $\alpha_1 = 0·5$.

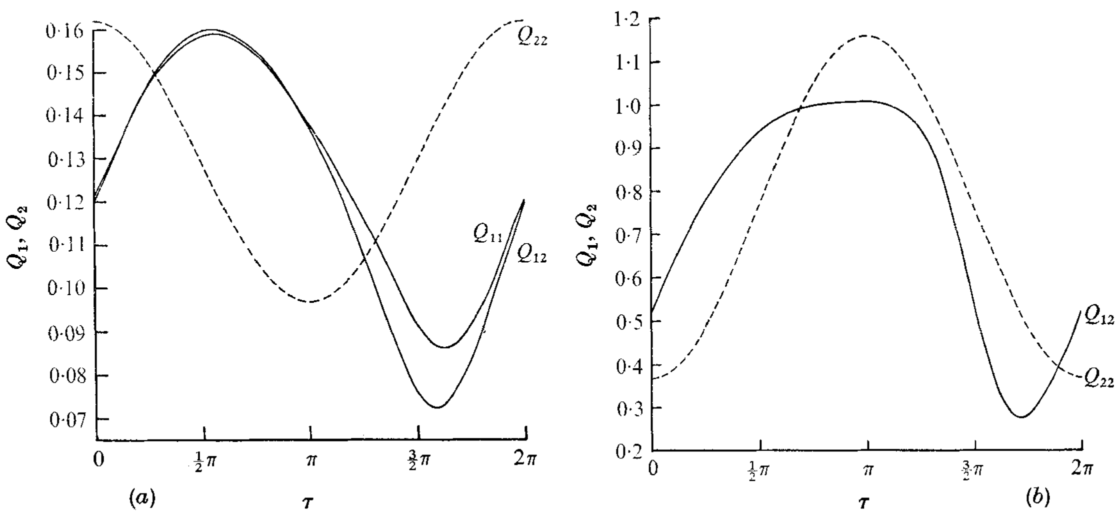

FIGURE 5. Graphs of $Q_1(x, \tau)$ and $Q_2(x, \tau)$ against τ as calculated from (2.9) and (3.32). Q_{pq} represents the q-term expansion of Q_p. ——, Q_1; – – –, Q_2. (a) $n = 0$, $\alpha_1 = 0·5$, $\epsilon_1 = 0·4$, $\sigma = 7·1$. (b) $n = \frac{1}{3}$, $\alpha_1 = 0·5$, $\epsilon_1 = 2·0$, $\sigma = 0·72$.

better throughout the cycle. In all other cases, the overlap is as poor as for $n = 0$, and when $n = 1$, $\sigma = 7·1$ overlap can in no sense be said to occur.

Part of the reason for the poor agreement in general between the two heat-transfer expansions may be that the large-ϵ_1 expansion (3.32) has not been extended to include second harmonic terms, which would appear at $O(\gamma_1^4)$, so that there is little flexibility in adjusting the phase of this expansion. However, since even the $O(\gamma_1^3)$ terms appear to decrease the accuracy at overlap values of ϵ_1, this cannot be the complete explanation. The principal reason for poor overlap is probably associated with the dependence on σ. For instance, consider the case $n = 0$, $\sigma = 0·72$, where overlap appears to occur for $\epsilon_1 = 5·0$. The heat-transfer expansion cannot in fact be accurate for $\epsilon_1 > 0·6$, because the velocity field used in the heat equation was calculated from the small-ϵ_1 stream-function expansion of § 2, shown above to be inaccurate for $\epsilon_1 > 0·6$. Similarly, in the case $n = 0$, $\sigma = 7·1$ (apparent overlap at $\epsilon_1 = 0·4$: see figure 5(a)) the large-ϵ_1 heat-transfer expansion will be inaccurate for $\epsilon_1 < 0·6$. In the one case where heat-transfer and skin-friction overlap occurs at the same value of ϵ_1 ($n = \frac{1}{3}$, $\sigma = 0{\cdot}72$, $\epsilon_1 = 2{\cdot}0$), agreement of phase as well as maximum and amplitude is achieved (see above and figure 5(*b*)).

A further possible reason for poor overlap is the presence of unknown multiples of the unsteady eigenfunctions discussed in §3 and the appendix. It is not possible to assess their importance in any given case without numerical solution of the boundary-layer equations. Such a solution was performed for the small amplitude flat-plate viscous boundary layer by Ackerberg & Phillips (1972). They confirmed the good overlap between the small- and large-ϵ_1 expansions for the amplitude and phase of the skin-friction variations, but noticed some discrepancy, in the overlap region, between the exact solution and the composite expansion for the phase. They then derived both the expansions and the exact numerical solution for the reduced volumetric flux, which is proportional to the unsteady part of

$$\lim_{\alpha_1 \to 0} \left[\frac{1}{\alpha_1}\int_0^\infty \{U(x,t) - u(x,y,t)\}\,dy\right],$$

and is a quantity which depends on the complete velocity profile, not merely its slope at the wall. Their results show that the exact solution for the amplitude and phase of this quantity, as functions of ϵ_1, oscillate vigorously about the large-ϵ_1 expansion for values of ϵ_1 up to at least 10 (whereas their skin-friction overlap value was 1·6). Thus for this quantity the asymptotic expansion is inaccurate. They attribute the oscillatory behaviour of the exact solution to the appearance of large multiples of the unsteady eigenfunctions already referred to. The above is a salutory reminder that good overlap between two asymptotic solutions for one part of the solution to a problem (here the skin friction) neither implies good overlap for the rest of the problem (e.g heat transfer) nor guarantees that either of them is close to the exact solution in the region of overlap.

The above results can be summarized as follows. For values of ϵ_1 above the chosen overlap value, say, $\epsilon_S(n)$ ($\approx 0{\cdot}6$ for $n = 0$, $2{\cdot}0$ for $n = \frac{1}{3}$, $6{\cdot}0$ for $n = 1$), the four-term expansion (3·27) is an accurate representation of the skin friction throughout the cycle. For $\epsilon_1 < \epsilon_S(n)$ the first two terms of the expansion (2.6) yield the most uniformly accurate expansion available for the skin friction, and it agrees well with the large-ϵ_1 expansion at $\epsilon_1 = \epsilon_S(n)$. Overlap between the small- and large-ϵ_1 expansions for heat transfer is not good. The values at which the two-term expansions (in each case the most uniformly accurate) agree most closely, i.e. the values of $\epsilon_H(n, \sigma)$ from table 5, depend markedly on the Prandtl number σ as well as on n. In any given case we should not use the small-ϵ_1 expansion for $\epsilon_1 > \min(\epsilon_S, \epsilon_H)$, nor the large-$\epsilon_1$ expansion for $\epsilon_1 < \max(\epsilon_S, \epsilon_H)$.

For some values of ϵ_1 and α_1 covered by this theory, the skin friction is negative over part of the cycle (see, e.g. figure 2(*b*)). For a given α_1, there is a value of ϵ_1, say ϵ_{1R} (corresponding to a given value of x), for which the minimum skin friction is zero; as ϵ_1 (and hence x, for $0 \leqslant n < 1$) increases above ϵ_{1R}, so the skin friction is negative over an increasing proportion of the cycle. During those parts of the cycle, the velocity profiles will resemble 'separated' velocity profiles, with a region of reversed flow near the wall. There has in the past (see e.g. Moore 1957) been considerable discussion on whether this phenomenon is the same as steady

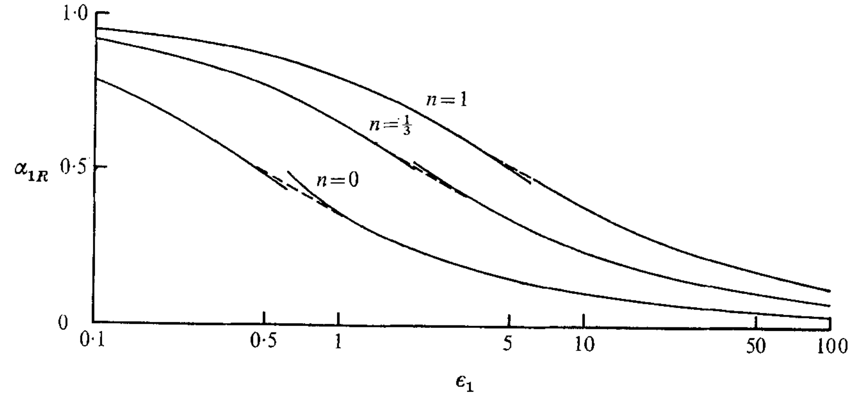

FIGURE 6. Graphs of the value of the amplitude parameter $\alpha_{1R}(\epsilon_1)$ at which skin friction at the wall reverses for a given value of ϵ_1, plotted in the three cases $n = 0, \frac{1}{3}, 1$. The broken lines have been inserted to provide a smooth join between the small- and the large-ϵ_1 expansions.

separation, when the boundary-layer approximation breaks down, but it seems clear that, if the separation streamline makes a sufficiently acute angle with the wall, there is no inconsistency in the use of boundary-layer theory. In our case, with $0 \leqslant n < 1$, it can be verified that $|v| \ll |u|$ at all points near the point of zero shear, and the boundary-layer approximation is quite consistent. During each cycle, the point of zero shear moves up the wall from $x = \infty$ (with a velocity which is initially infinite) to a point given by $\epsilon_1(x) = \epsilon_{1R}$, and then moves off to infinity again. If we were considering the case $n < 0$, where *steady* separation occurs, the unsteadiness would have a profound effect.

The question of whether a given point on the wall suffers a reversal of shear, for an outer flow of a certain amplitude, can be answered for each value of ϵ_1 (i.e. x) by calculating that value of α_1, say $\alpha_{1R}(\epsilon_1)$, for which the minimum skin friction is zero. These calculations have been made for the cases $n = 0$, $\frac{1}{3}$ and 1, and the results are shown in figure 6 (the small- and large-ϵ_1 expansions are joined near overlap by a broken line).† As n increases, so does the value of α_{1R} for a given ϵ_1: e.g. for $\epsilon_1 = 1$, $\alpha_{1R} = 0{\cdot}36$ for $n = 0$, $0{\cdot}65$ for $n = \frac{1}{3}$, and $0{\cdot}80$ for $n = 1$. This is of course expected, since an increase in n is equivalent to an increasingly favourable pressure gradient, which can be countered near the wall only by an increase in the maximum adverse pressure gradient available during the oscillation, i.e. by an increase in α_1. Note that the case $n = 1$ is different from the others in that ϵ_1 is independent of x. Thus if there is skin-friction reversal at one point of the wall $y = 0$, there is simultaneous reversal at every point of the wall.

Finally, it is pertinent to ask whether this theory can ever be valid for flows in which the free stream reverses, i.e. $\alpha_1 > 1$. We cannot have outer flow reversal near the leading edge (small x, or small ϵ_1) when $n < 1$, because the initial condition

† For $n = 0$, α_{1R} can be readily calculated from the large-ϵ_1 expansion to be

$$\alpha_{1R}(\epsilon_1) = \alpha_2\gamma_1\{1+(1-\tfrac{5}{16}\alpha_2\gamma_1^3)^2\}^{-\frac{1}{2}}, \tag{4.1}$$

but no such simple analytic expression is available from the small-ϵ_1 expansion or for other values of n.

that $u(0,y,t)$ be prescribed could no longer be applied. Mathematically, this follows from the breakdown both of the transformation (2.1) if $U(x,\tau) < 0$ and of the small-ϵ_1 expansion when $V(\tau)$ is zero (see equations (2.4)). This breakdown also occurs for $n = 1$, when there is no leading edge as such, and follows from the non-existence of steady stagnation-point flow away from a wall, so that no quasi-steady solution can be found. For large values of ϵ_1 (i.e. large x, for $n < 1$), however, the expansion of §3 is formally valid for all values of α_1.† This has been noticed before (Lin 1956; Gersten 1965) and follows from the complete uncoupling of the mean and oscillatory parts of the flow. In the stagnation-point case ($n = 1$) this means that a solution can be found with $\alpha_1 > 1$ for all x, but in all other cases the need to find a solution at small x enforces the restriction $\alpha_1 < 1$.

I acknowledge with gratitude my indebtedness to Professor Sir James Lighthill, at whose instigation I first became interested in unsteady boundary layers. I am grateful for financial support to the Royal Society, the Merchant Taylors' Company, the Nuffield Foundation and the Medical Research Council.

Appendix. On the existence of eigensolutions

Steady eigenfunctions

We wish to discuss the existence of eigensolutions to the problem posed by equations of the form

$$F'''' + (n+1)f_0F''' + \lambda f_0'F' + (1-3n-\lambda)f_0''F = 0, \qquad \text{(A 1)}$$

where $f_0(\eta)$ is given by (2.3) and its associated boundary conditions, subject to the boundary conditions

$$F(0) = F'(0) = 0, \quad F'(\eta) \overset{e}{\to} 0 \quad \text{as} \quad \eta \to \infty. \qquad \text{(A 2)}$$

The form (A 1) includes all equations (2.5) for the functions $f_{mk}(\eta)$ occurring in the small-ϵ_1 expansion, in which case

$$\lambda = -2[m+n(2-m)]; \qquad \text{(A 3)}$$

it also includes the equations (3.12), (3.20), etc., for the functions of integration $F_m(\eta)$ in the large-ϵ_1 expansion, in which case

$$\lambda = m - n(m+4). \qquad \text{(A 4)}$$

We are principally concerned with the range $0 \leqslant n \leqslant 1$, and m is always a non-negative integer. Initially the analysis closely follows that of Libby & Fox (1963), who examined the case $n = 0$ in detail.

We first notice that $F = f_0'$ is one solution of (A 1), and therefore make the transformation $F = f_0'G$, $H = G'$ to obtain a second-order equation for H:

$$f_0'H'' + [3f_0'' + (n+1)f_0f_0']H' + [(\lambda+6n)f_0'^2 - 6n - (n+1)f_0f_0'']H = 0. \qquad \text{(A 5)}$$

We shall need to know the asymptotic form as $\eta \to \infty$ of the three independent solutions of (A 1). The asymptotic form for f_0' is

$$f_0' \sim 1 + \hat{\alpha}_1\hat{\eta}^{-(1+5n)/(1+n)}\exp[-\tfrac{1}{2}(n+1)\hat{\eta}^2],$$

† Although it will be useful only if successive terms of equation (3.27) decrease.

where $\hat{\eta} = \eta - \beta$ and $\beta = \lim_{\eta\to\infty} (\eta - f_0)$ and is given in table 1. The general solution of (A 5) can then be shown to have asymptotic form

$$H \sim \hat{\alpha}_2 \hat{\eta}^{-\lambda/(n+1)} + \hat{\alpha}_3 \hat{\eta}^{\lambda/(n+1)-1} \exp\left[-\tfrac{1}{2}(n+1)\hat{\eta}^2\right], \tag{A 6}$$

from which the general asymptotic form for F' can readily be derived. Exponential decay is assured if $\hat{\alpha}_2$ is zero, and since the problem for H is a Sturm–Liouville problem, we anticipate an infinite number of real discrete values of λ for a given n (or of n for a given λ) for which exponentially decaying eigenfunctions exist. The reality of the eigenvalues is proved in the usual way.

The first general result about the eigenvalues is obtained if we multiply (A 5) by $Hf_0'^3 \exp\left\{(n+1)\int_0^\eta f_0(t)\,dt\right\}$ and integrate from 0 to ∞. After one integration by parts, we have

$$-[f_0'^3 \exp\{\ \}HH']_0^\infty + \int_0^\infty \{f_0' H'^2 + [6n(1-f_0'^2) + (n+1)f_0 f_0'']H^2\} \times f_0'^2 \exp\{\ \}\,d\eta = \lambda \int_0^\infty f_0'^4 H^2 \exp\{\ \}\,d\eta,$$

where $\{\ \} \equiv \left\{(n+1)\int_0^\eta f_0(t)\,dt\right\}$. The asymptotic forms with $\hat{\alpha}_2 = 0$ show that if H is an eigenfunction then the first term of the above equation is zero. The integrands of both the other terms are greater than or equal to zero for all η, so no non-trivial eigensolution can exist if $\lambda \leqslant 0$. Thus no eigensolutions occur in the small-ϵ_1 expansion, where λ is given by (A 3); this is of course expected on physical grounds. In the large-ϵ_1 expansion, where λ is given by (A 4), we see that eigensolutions cannot exist if

$$m \leqslant 4n/(1+n). \tag{A 7}$$

The second general result comes from considering the first positive eigenvalue λ_0. The corresponding eigensolution F_0 will be such that F_0' has no zero between $\eta = 0$ and $\eta \to \infty$; without loss of generality $F_0''(0)$ may be taken to be strictly positive, so that F_0' and F_0 are greater than or equal to zero for all η (recall that f_0, f_0' and f_0'' are greater than or equal to zero for all η). We integrate (A 1) directly from 0 to ∞, applying the given boundary conditions, and obtain the following pair of equations:

$$2(\lambda_0 + n - 1)\int_0^\infty f_0' F_0'\,d\eta = F_0''(0) + (\lambda_0 + 3n - 1)F_0(\infty), \tag{A 8a}$$

$$-2(\lambda_0 + n - 1)\int_0^\infty f_0'' F_0\,d\eta = F_0''(0) - (\lambda_0 - n - 1)F_0(\infty). \tag{A 8b}$$

In each case, the integrand on the left-hand side is strictly positive, and, if (*a*) $\lambda_0 \geqslant 1 - 3n$ or (*b*) $\lambda_0 \leqslant 1 + n$, the right-hand side is also strictly positive. Hence no solution can exist if, in addition, (*a*) $\lambda_0 \leqslant 1 - n$ or (*b*) $\lambda_0 \geqslant 1 - n$. Therefore the first eigenvalue cannot lie in the range

$$1 - 3n \leqslant \lambda_0 \leqslant 1 + n. \tag{A 9}$$

Now, if $n \geqslant \frac{1}{3}$, the left-hand end of this range is less than or equal to zero, and no eigenvalues less than or equal to zero are possible, so in that case no eigensolutions exist if $\lambda \leqslant 1+n$. From (A 4) this means that, with $n \geqslant \frac{1}{3}$, no eigensolutions are possible with $m \leqslant (1+5n)/(1-n)$. In the case $n = 1$, this means that no eigensolutions exist for any m; the large-ϵ_1 expansion is fully determined. In the case $n = \frac{1}{3}$, we can rule out eigensolutions only for $m \leqslant 4$. In the last case of interest to us, $n = 0$, equation (A 7) shows that no eigensolutions exist for $m \leqslant 0$ and (A 9) shows that the first eigensolution is not $m = 1$, but this type of analysis takes us no further. However, Stewartson (1957) has shown that $m = 2$ is an eigenvalue and that, probably, no other eigenvalue is an integer. Libby & Fox (1963) verified numerically that $m = 2$ is the first eigenvalue, and that the next nine are not integers. Thus of all the equations of the form (A 1) occurring in the main body of this paper, the only one possessing an eigensolution is (3.12) for $F_2(\eta)$, in the case $n = 0$; the eigenfunction is (Stewartson 1957)

$$F_2(\eta) = f_0 - \eta f_0'. \tag{A 10}$$

Next we turn to those ordinary differential equations occurring in the analysis of the heat equation. They are of the form

$$(1/\sigma)\,G'' + (n+1)f_0 G' + \mu f_0' G = 0, \tag{A 11}$$

subject to boundary conditions $G(0) = G(\infty) = 0$ (cf. Fox & Libby 1964). In the small-ϵ_1 expansion,

$$\mu = -2m(1-n) \tag{A 12}$$

and in the large-ϵ_1 expansion,

$$\mu = m(1-n), \tag{A 13}$$

where, as before, $0 \leqslant n \leqslant 1$ and m is a non-negative integer. The asymptotic form for G is

$$G \sim \hat{\alpha}_1 \hat{\eta}^{-\mu/(n+1)} + \hat{\alpha}_2 \hat{\eta}^{\mu/(n+1)-1} \exp\left[-\tfrac{1}{2}\sigma(n+1)\,\hat{\eta}^2\right].$$

In a manner similar to that used above with (A 5), we can show that the eigenvalues are real and positive. Thus, when μ is given by (A 12) there are no eigensolutions, and when μ is given by (A 13) there are none if $n \geqslant 1$. When $\mu > 0$, the first eigenfunction G_0 can without loss of generality be taken to be everywhere greater than or equal to zero, with $G_0'(0) > 0$. An integration of (A 11) gives

$$(\mu_0 - 1 - n)\int_0^\infty f_0' G_0\, d\eta = \frac{1}{\sigma} G'(0) > 0,$$

so the first eigenvalue cannot be less than or equal to $1+n$, and hence no eigenvalue can be less than or equal to $1+n$. Thus when μ is given by (A 13), we see that there are no eigenvalues if

$$m \leqslant (1+n)/(1-n). \tag{A 14}$$

When $n = \frac{1}{3}$, this rules out eigensolutions for $m \leqslant 2$, and when $n = 0$ they are ruled out only for $m \leqslant 1$. The eigenvalues of (A 11) will depend on σ, and will not in general be integers [Fox & Libby (1964) showed that none of the first 10 are integers when $\sigma = 1$]. In the cases $m = 2, 3$ $(n = 0)$ and $m = 3$ $(n = \frac{1}{3})$, for the two values of σ used in this paper, the numerical solution of (3.30) was examined to see if all values of $G'_m(0)$ led to a solution satisfying the boundary condition at infinity, which would indicate the presence of an eigenfunction. In no case did such behaviour occur, although in the case $n = 0$ the solution for G_2 is indeterminate, because F_2 contains an eigenfunction.

Unsteady eigenfunctions

Consider first the form of such a solution in outer variables. We seek an asymptotic solution of (3.4), as $x \to \infty$, of the form

$$\tilde{\psi} = \hat{\psi} + \phi,$$

where $\hat{\psi}$ is the algebraic asymptotic expansion (3.7), analysed in detail in §3, and ϕ is expected to be exponentially small as $x \to \infty$. If we therefore neglect nonlinear terms in ϕ, and retain only the largest terms in the coefficients as $x \to \infty$, we obtain the following partial differential equation for ϕ:

$$\phi_{\eta\tau} = Kx^n\{f_0''\phi_x - (\alpha_1 \sin\tau + f_0')\phi_{x\eta}\},$$

where $K = \gamma_1^2 x^{-(1-n)} = U_0(x)\,x^{-n}/\omega$. The solution must satisfy the usual (outer) boundary conditions in η, be periodic in τ and decay exponentially as $x \to \infty$. Possible solutions in which the x dependence is separable, and exponential, can be found, and the substitution

$$\phi = \exp\left\{-\lambda^2\left[\frac{x^{1-n}}{K(1-n)} + \alpha_1\cos\tau\right]\right\} P(\eta, \tau)$$

$(n < 1)$ leads to the following equation for P:

$$P_{\eta\tau} = \lambda^2(f_0' P_\eta - f_0'' P).$$

The general solution of this is not available, but the only acceptable separable solution which leads to a sensible matching condition for the inner expansion is

$$P = f_0'(\eta). \tag{A 15}$$

The corresponding function in inner variables is obtained by substituting $\Psi = \hat{\Psi} + \Phi$, where $\hat{\Psi}$ was obtained in §3, into (3.15) and linearizing in the same way. The substitution

$$\Phi = \exp\left\{-\lambda^2\left[\frac{x^{1-n}}{K(1-n)} + \alpha_1\cos\tau\right]\right\} Q(\zeta, \tau)$$

leads to a partial differential equation for Q which, after one integration, becomes

$$Q_{\zeta\zeta} - 2Q_\tau - 2\lambda^2\alpha_1 e^{-\zeta}\sin(\tau - \zeta)\,Q = \dot{A}(\tau), \tag{A 16}$$

where $\dot{A}(\tau)$ is the function of integration. The boundary conditions on Q are

$$Q(0,\tau) = Q_\zeta(0,\tau) = 0 \quad [Q_{\zeta\zeta}(0,\tau) = \dot{A}(\tau)], \quad Q(\zeta,\tau) \sim \alpha_2\zeta - \tfrac{1}{2}A(\tau) \quad \text{as} \quad \zeta \to \infty$$

(matching with (A 15)), and Q must be periodic in τ. Together with (A 16) these conditions constitute an eigenvalue problem for λ^2, for which I have been unable to prove that solutions do (or do not) exist. Note that the above form of solution is different from that found by Ackerberg & Phillips (1972) for the small-α_1 equations.

REFERENCES

ACKERBERG, R. C. & PHILLIPS, J. H. 1972 The unsteady laminar boundary layer on a semi-infinite flat plate due to small fluctuations in the magnitude of the free-stream velocity. *J. Fluid Mech.* **51**, 137.

FOX, H. & LIBBY, P. A. 1964 Some perturbation solutions in laminar boundary-layer theory. Part 2. The energy equation. *J. Fluid Mech.* **19**, 433.

GERSTEN, K. 1965 Heat transfer in laminar boundary layers with oscillating outer flow. *AGARDograph*, **97**, 423.

GHOSH, A. 1961 Contribution a l'étude de la couche limite laminaire instationnaire. *Publ. Sci. de la Ministère de l'Air*, no. 381.

GIBELLATO, S. 1955 Strato limito attorno ad una lastra piana investita da un fluido incompressibile dotato di una velocità che è somma di una parte constante e di una parte alternata. *Atti Accad. Torino*, **89**, 180 and **90**, 1.

GIBSON, W. E. 1957 Unsteady boundary layers. Ph.D. dissertation, M.I.T.

ISHIGAKI, H. 1970 Periodic boundary layer near a two-dimensional stagnation point. *J. Fluid Mech.* **43**, 477.

LAM, S. H. & ROTT, N. 1960 Theory of linearized time-dependent boundary layers. *Cornell Univ. GSAE Rep.* AFOSR TN-60-1100.

LIBBY, P. A. & FOX, H. 1963 Some perturbation solutions in laminar boundary-layer theory. Part 1. The momentum equation. *J. Fluid Mech.* **17**, 433.

LIGHTHILL, M. J. 1954 The response of laminar skin friction and heat transfer to fluctuations in the stream velocity. *Proc. Roy. Soc.* A **224**, 1.

LIN, C. C. 1956 Motion in the boundary layer with a rapidly oscillating external flow. *Proc.* IX^e *Int. Cong. Appl. Mech.*, Brussels, **4**, 155.

MOORE, F. K. 1951 Unsteady laminar boundary layer flow. *N.A.C.A. Tech. Note*, no. 2471.

MOORE, F. K. 1957 Aerodynamic effects of boundary-layer unsteadiness. *Proc. 6th Anglo-Amer. Aero. Conf., Roy. Aero. Soc.*, Folkestone, p. 439.

PEDLEY, T. J. 1972 On the forced heat transfer from a hot film embedded in the wall in two-dimensional unsteady flow. *J. Fluid Mech.* **55**, 329.

ROSENHEAD, L. (ed.) 1963 *Laminar Boundary Layers.* Oxford University Press.

ROTT, N. & ROSENZWEIG, M. L. 1960 On the response of the laminar boundary layer to small fluctuations of the free-stream velocity. *J. Aerospace Sci.* **27**, 741.

STEWARTSON, K. 1957 On asymptotic expansions in the theory of boundary layers. *J. Math & Phys.* **36**, 173.